Fundamental groups of spaces of trigonal curves

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Outline

Introduction

- Braid group = the fundamental group of the space of complex polynomials with distinct roots.
- Natural generalizations of the braid group are the fundamental groups of spaces of nonsingular hypersurfaces on algebraic varieties.
- It is the case when we try to generalize to the bigger dimensions the method of braids in the theory of real algebraic curves.

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Summary

- For the space of $PGL(2, \mathbf{C})$-orbits of the space of complex trigonal curves on Hirzebruch surface Σ_k, a stratification and a cell structure of each stratum has been constructed using the Lyashko-Looijenga mapping. The cell structure is described via Grothendiek's *dessins d'enfants*.
- For the space of nonsingular complex trigonal curves on the Hirzebruch surface Σ_k and for its subspace of the curves with the simple roots of the discriminant of the curve equation, the fundamental groups (for $k = 1$) and their images in the spherical braid group (for any k) have been calculated.

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Definitions

- $q : \Sigma_k \to \ P^1$ is the (complex) Hirzebruch surface, i.e. a rational ruled surface with the exceptional section s, $s^2 = -k < 0$. The fibers of q are *vertical*.
- A *trigonal curve* is a curve $A \subset \Sigma_k$ disjoint from $s(P^1)$, with the restriction $q : A \to P^1$ of degree 3 and being not a cube of a rational curve.
- Contraction:
$$\begin{array}{ccl} s(P^1) & \mapsto & \text{point} \\ \cap & & \cap \\ \Sigma_k & \to & \mathbf{P(1,1,k)} \text{– the weighted projective plane with the} \\ \cup & & \cup \qquad \text{coordinates } x_0, x_1, y \text{ of weights } 1, 1, k. \\ A & \to & y^3 + b(x_0, x_1)y + w(x_0, x_1) = 0, \end{array}$$
where b, w are homogeneous polynomials of degrees $2k$, $3k$ and $y = \infty$ is the image of s.
- b, w are determined by A uniquely up to the transformation
$$(b, w) \mapsto (t^2 b, t^3 w),\ t \in \mathbf{C}^*.$$
So the set of all trigonal curves on Σ_k is the weighted projective space $\mathbf{P(2, \ldots, 2, 3, \ldots, 3)}$ of complex dimension $5k + 1$.

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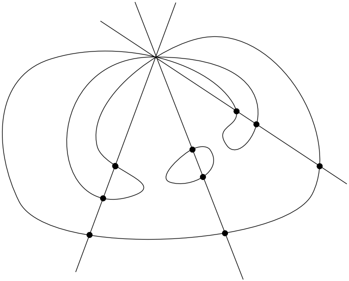

Figure : The pencil of lines gives a braid

Correspondence:

A curve with a singular point.	A surface of degree 5 with two singular points.
The pencil of lines centered in the point.	The pencil of planes centered in the line through the points.
A braid in 3 strings.	A family of trigonal curves.

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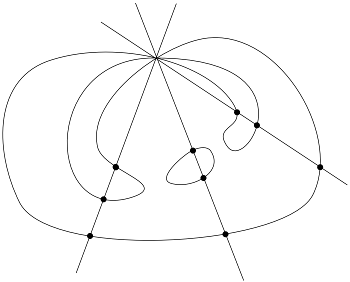

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Historical remarks

- In 1999, the method we are using here (a modification of Grothendiek's dessin d'enfant) was offered in
S. Yu. Orevkov. Riemann existence theorem and construction of real algebraic curves//*Annales de la Faculté des Sciences de Toulouse, 2003, 12(4), p. 517-531*,
and applied to real trigonal curves also in
Degtyarev A., Itenberg I., Kharlamov V. On deformation types of real elliptic surfaces. // Amer.J.Math. 130(2008), no.6, p.1561-1627,
where equivariant deformations of real trigonal curves on a ruled surface over a base of any genus were studied and an explicit description of the deformation classes of M- and $(M-1)$-curves were obtained.
- This method was developed, both for complex and real trigonal curves, in the recent book
Degtyarev A. Topology of algebraic curves. An approach via dessins d'enfants. *de Gruyter Studies in Mathematics, 44. Walter de Gruyter* & Co., Berlin, 2012. xvi+393pp.

j-invariant of the curve $A: y^3 + b(x)y + w(x) = 0$

- $d = 4b^3 + 27w^2$ is the discriminant in y of the curve equation.
  $j : \mathbf{P^1} \to \mathbf{P^1}, \mathbf{j = 4b^3/d = 1 - 27w^2/d}$ is *the j-invariant of the curve A*.
- $Gr(j) \subset \mathbf{P^1} \times \mathbf{P^1}$ is the curve $4b^3(x)y_0 - d(x)y_1 = 0$,
  $x = [x_0 : x_1], y = [y_0 : y_1] \in \mathbf{P^1}$. It's the union of the graph of the function j and the lines $g.c.d.(b^3, d) = 0$ with the corresponding multiplicities.
  A and $Gr(j)$ uniquely determine to each other.

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- $Jconst \subset \mathbf{P}(\mathbf{2}, \ldots, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{3})$ is the set of trigonal curves on Σ_k with the constant j-invariant.
  $Jconst = \{y^3 + b(x)y + w(x) = 0 | b = \lambda a^2, w = a^3\} \cong P^1 \times P^k$.
  Denote $\mathbf{P}(\mathbf{2}, \ldots, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{3}) \setminus Jconst$ by $Trig_k$ and $\mathbf{P}(\mathbf{2}, \ldots, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{3})$ by $\overline{Trig_k}$.
- $Trig_k$ is nonsingular.

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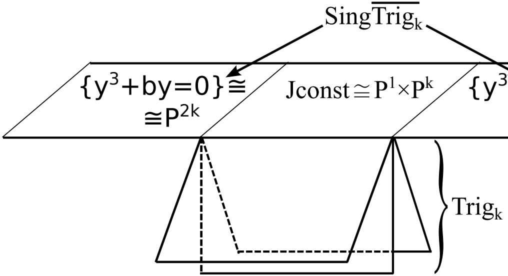

Figure : The space $\overline{Trig_k}$ of trigonal curves on Σ_k

- $PGL(2, \mathbf{C}) = AutP^1$ acts on the arguments of the polynomials $b(x)$, $w(x)$ in the curve equation. So it acts in $Trig_k$ and in the space $\{Gr(j) | j = 4b^3/d = 1 - 27w^2/d\}$.

- *The closure of the orbit of a curve $Gr(j)$ consists of this orbit and the set $\{$triple of lines $(l_0x_0 + l_1x_1)^p(c_0x_0 + c_1x_1)^{6k-p}(b_1(\alpha)y_0 - d_1(\alpha)y_1) = 0 | [l_0 : l_1], [c_0 : c_1] \in \mathbf{P}^1$, $\alpha = [a_0 : a_1]$ is a root of degree p of g.c.d.(b^3, d) $b_1 = 4b^3/(a_0x_0 - a_1x_1)^p, d_1 = d/(a_0x_0 - a_1x_1)^p$ $\}$*

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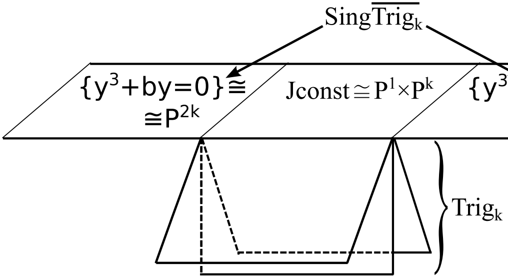

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- $Trig_k/PGL(2, \mathbf{C})$ is second countable.
- A limit of a sequence in $Trig_k/PGL(2, \mathbf{C})$ is unique.

Corollary

$Trig_k/PGL(2, \mathbf{C})$ is a Hausdorff space.

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The projection $rd : Trig_k \to Trig_k/PGL(2,\mathbf{C})$

- $A \in Trig_k$ is *symmetric*, if $\exists g \in PGL(2,\mathbf{C}) : g(A) = A$. $Symm$ is the set of all symmetric curves.

Theorem

Let $T_3 \subset Trig_k \setminus Symm$ be the set of curves with the total number of simple roots of the polynomials $b(x)$, $w(x)$, $d(x)$ being not less than 3. Then the projection $rd : T_3 \to T_3/PGL(2,\mathbf{C})$ is a locally trivial principal $PGL(2,\mathbf{C})$-fibration and $rd(T_3)$ is manifold.

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Trichotomic graph of a trigonal curve

$A:\ y^3+b(x)y+w(x)=0$

S. Yu. Orevkov. Riemann existence theorem and construction of real algebraic curves//*Annales de la Faculté des Sciences de Toulouse, 2003, 12(4), p. 517-531*:

A trigonal curve up to the transformation $(x,y)\mapsto(x,\lambda y)$, $\lambda\neq 0$	$\leftrightarrow$	the colored graph $\Gamma(j)=j^{-1}(\mathbb{R}P^1)$ on S^2

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Trichotomic graph $\Gamma(j)$:

For A with $j=const$ we have no $\Gamma(j)$ and use only $Gr(j)$.

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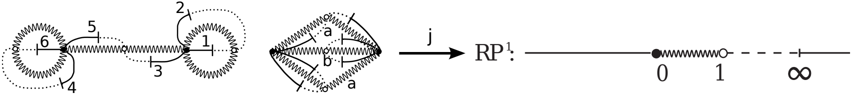

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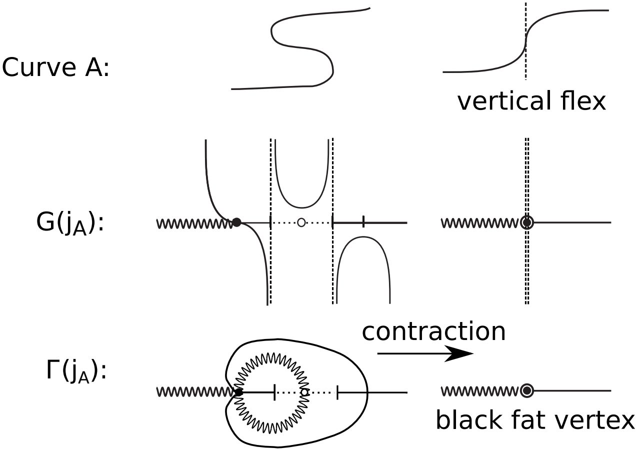
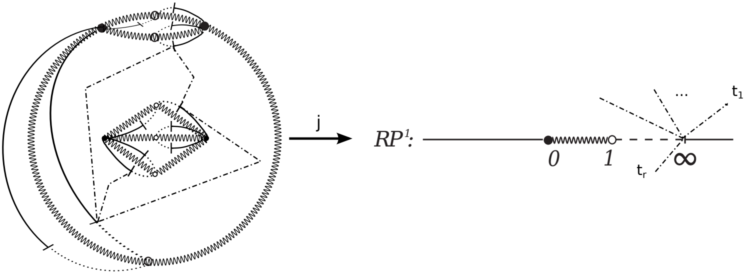
Fat vertices

A fat vertex corresponds to a common root x of b, w. There are 3 kinds of fat vertices: ●-vertex, ○-vertex and waved-vertex depending on $3mult_x(b(x)=0) >, <, = 2mult_x(w(x)=0)$. For ○-vertex and waved-vertex x is a singular point of the trigonal curve.



Tetratomic graph of a trigonal curve

We orient the faces of a trichotomic graph Γ in the chessboard order and obtain a *tetratomic graph* $\mathrm{T}\Gamma$ by a partition of every face of Γ on simply connected parts; the boundary of every part can be glued into a wedge of colored $\mathbb{R}\mathbf{P}^1$ and segments which number r_+ or r_- depends only on the orientation of the face of Γ, the center of the wedge being ∞.



The *tetratomic graph of a curve* $A(j) \in Trig_k$:
$t_1, \ldots, t_r$ are all the imaginary critical values of j,
$St(j)$ is the star in $\mathbf{CP}^1$ with the center at ∞ and the ray ends in t_i (a ray may contain another one),
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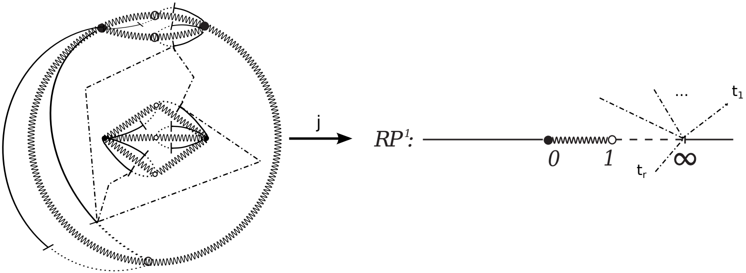

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Riemann's data of a trigonal curve

- The pair $(\mathrm{T\Gamma(j)}, \{\mathrm{t_1, \ldots, t_q}\})$ with $\{t_1, \ldots, t_q\}$ being the set of all critical values of j is the *Riemann's data* of the curve $A(j)$ (see **S.K.Lando, A.K.Zvonkin.** Graphs on Surfaces and Their Applications, Springer-Verlag, 2004.)
- Let $\mathrm{T\Gamma}$ be a tetratomic graph with ∘-vertices of valence $0 \bmod 4$, •-vertices of valence $0 \bmod 6$ and fat-vertices of valence $0 \bmod 2$. Due to Lando and Zvonkin $T\Gamma$ *with the set* $\{0, 1, \infty, t_1, \ldots, t_r\} \subset \mathbf{C}$ *can be presented as the Riemann's data of a trigonal curve* $A \in Trig_k$ *unique up to the action of* $PGL(2, \mathbf{C})$.

Corollary

The set of Riemann's data of trigonal curves can be identified with the quotient space $RD_k = Trig_k / PGL(2, \mathbf{C})$.

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The Lyashko-Looijenga mapping

- For a rational function $f : \mathbf{CP^1} \to \mathbf{CP^1}$ the Lyashko-Looijenga mapping is

$$LL(f) = (t - t_1)^{l_1} \dots (t - t_r)^{l_r},$$

where $t_1, \dots, t_r$ are all the finite critical values of f,
$l_i = \sum_{x \in f^{-1}(t_i)} mult_x(f'(x) = 0)$.

Lemma

Up to a multiplicative constant, $LL(P(x)/Q(x))$ is the discriminant of the polynomial $P(x) - tQ(x)$.

- The homogeneous variant of the Lyashko-Looijenga mapping:
$LL([P(x) : Q(x)]) = discr_x(t_0 P(x) - t_1 Q(x)) \in \mathbf{CP^{2n-2}}$ where
$n = \deg P = \deg Q$.
For a trigonal curve $A(j)$ let $LL(A) = LL(j)$.

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$$LL(f) = (t - t_1)^{l_1} \dots (t - t_r)^{l_r},$$
where $t_1, \dots, t_r$ are all the finite critical values of f,
$l_i = \sum_{x \in f^{-1}(t_i)} mult_x(f'(x) = 0)$.

Lemma

Up to a multiplicative constant, $LL(P(x)/Q(x))$ is the discriminant of the polynomial $P(x) - tQ(x)$.

- The homogeneous variant of the Lyashko-Looijenga mapping:
$LL([P(x) : Q(x)]) = discr_x(t_0 P(x) - t_1 Q(x)) \in \mathbf{CP^{2n-2}}$ where
$n = \deg P = \deg Q$.
For a trigonal curve $A(j)$ let $LL(A) = LL(j)$.

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- $Trig_k$ has a stratification depending on the $\deg j = 6k - \deg g.c.d.(b^3, w^2)$. The adjacency of strata is described in terms of $Gr(j)$:

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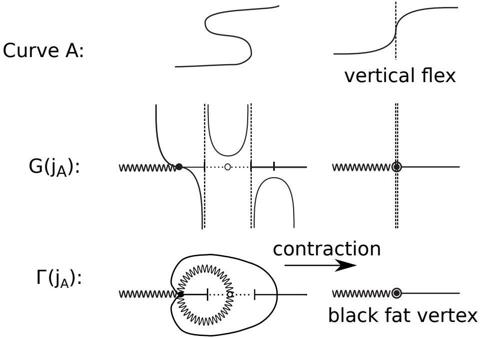

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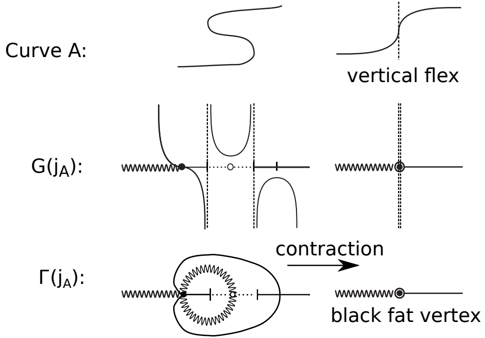

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A cell structure

- $\deg g.c.d.(b^3, w^2) = p$, $T_p \subset Trig_k$ is the corresponding stratum, $P_{r,s} \subset LL(T_p) = \{t^{4k-2p}(t-1)^{3k-p}(t-t_1)\dots(t-t_{5k-2p-2})\}$ is the subset of polynomials with the following condition: the number of different roots of the polynomial including $0, 1$ is r, the number of different arguments of the roots is s (the argument of 0 being π).
  $P_m = \bigcup_{r+s-4=m} P_{r,s}$.
- Any connected component of P_m is convex since it's determined by linear equations and inequalities; thus it's an open cell of dimension m.
  The collection of the components of all the sets P_m is an open sell partition of $LL(T_p)$.
- $LL : T_p \xrightarrow{rd} RD_k \xrightarrow{\overline{LL}} \mathbf{CP^{12k-5p-2}}$
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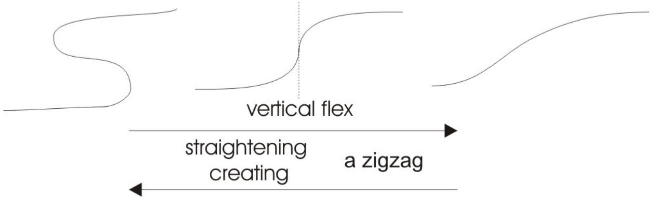
Generic and almost generic trigonal curves

A nonsingular curve $A \in Trig_k$ is *almost generic* if it's nonsingular and has no vertical flexes:



In particular $\deg j = 6k$, $\infty \in \mathbf{CP}^1$ is a regular value of j, and the roots of the equations $j(x) = 0$ and $j(x) = 1$ have respectively multiplicities $0 \bmod 3$ and $0 \bmod 2$. If these multiplicities equals respectively 3 and 2 then an almost generic curve is *generic*.

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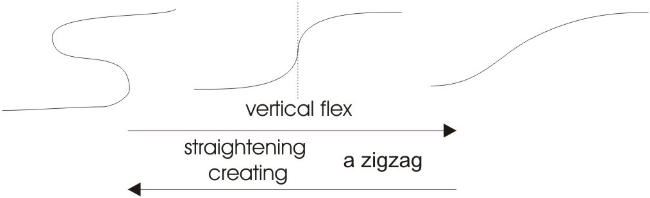

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Dual partition of the space of Riemann's data

- Let $NSing_k \subset Trig_k$ be the space of nonsingular trigonal curves and $AlGen_k \subset NSing_k$ be the subspace of almost generic curves.
- The closure of a cell of the space RD_k is convex $\Rightarrow$ the removal of a cell which points are singular curves allows to contract the adjacent cells to their boundaries.
- There is a partition of the contracted RD_k dual to its cell partition.
- Let $Sk_2DualNSing_k$, $Sk_2DualAlGen_k$ be the 2-skeletons of the dual partitions of the spaces $NSing_k/PGL(2, \mathbf{C})$, $AlGen_k/PGL(2, \mathbf{C})$. They turn out to be cell complexes.
- $codim_{\mathbb{R}} Symm > 2$, so we may consider that $NSing_k$ and $AlGen_k$ have no symmetric curves.
- Since $\pi_1(PGL(2, \mathbf{C})) \cong \mathbf{Z}_2$, $\pi_2(PGL(2, \mathbf{C})) = 0$, using the exact sequences of the fiberings $rd : NSing_k \to NSing_k/PGL(2, \mathbf{C})$, $rd : AlGen_k \to AlGen_k/PGL(2, \mathbf{C})$ we can prove

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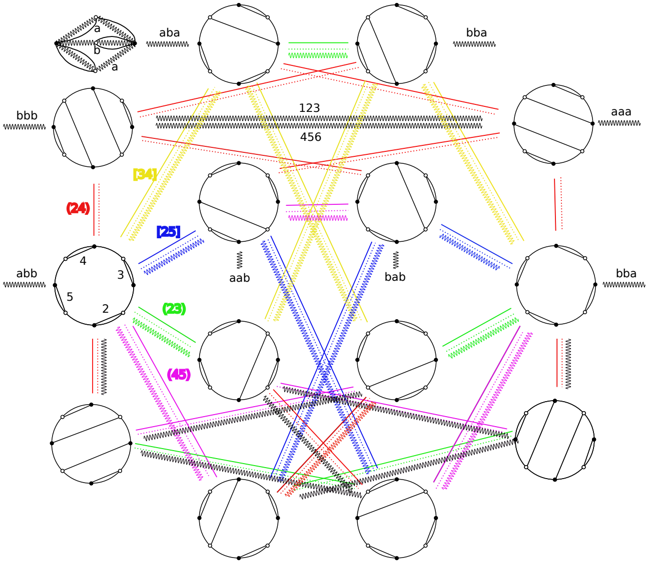

Figure : 2-skeleton of the dual partition of the space $AlGen_1/PGL(2,\mathbf{C})$ (homotopic loops are of the same color).

The fundamental group of the space of almost generic curves on Σ_1

$$0 \to \mathbf{Z}_2 \to \pi_1(AlGen_k) \to \pi_1(Sk_2DualAlGen_k) \to 0$$

Theorem

$\pi_1(Sk_2DualAlGen_1) =$
$\langle (23), (24), [25], [34], (45), (\tilde{23}), (\tilde{24}), [\tilde{25}], [\tilde{34}], (\tilde{45}), (123)(654);$
$[25]^2 = [34]^2 = 1, (23)(45) = (45)(23), [25][34] = [34][25], [\tilde{25}]^2 =$
$[\tilde{34}]^2 = 1, (\tilde{23})(\tilde{45}) = (\tilde{45})(\tilde{23}), [\tilde{25}][\tilde{34}] = [\tilde{34}][\tilde{25}], (24)(123)(654) =$
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The square brackets denote the elements of order 2.

The fundamental group of the space of nonsingular curves on Σ_1

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$\pi_1(Sk_2 DualNSing_1)$ is the quotient of $\pi_1(Sk_2 DualAlGen_1)$ by $(\tilde{23}) = (\tilde{45}) = 1$.

The braid group of sphere

- Let $C_n(S^2)$ be the configuration space of non-ordered sets of n distinct points of S^2.
  $\pi_1(C_n(S^2)) = \mathrm{H_n}$ is the braid group of sphere.
- Generators $\sigma_1, \dots, \sigma_{n-1}$ of $\mathrm{H_n}$:
- The relations between the generators: standard relations + an additional one $\sigma_1 \dots \sigma_{n-1}\sigma_{n-1} \dots \sigma_1 = 1$ (see **Faddel E., Van Buskirk J.** The braid groups of E^2 and S^2 // Duke Math. J., 1962, v. 29, p.243-258.)

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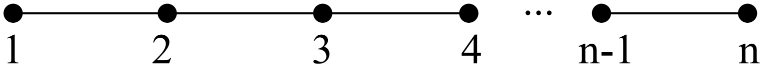

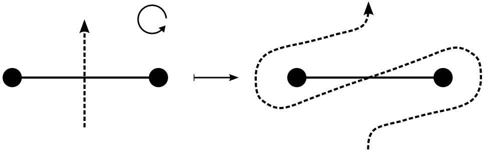

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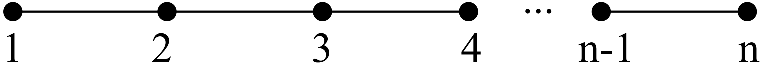

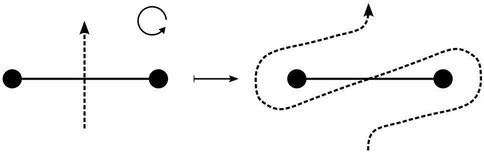

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A homomorphism to the braid group of sphere

- The mapping $AlGen_k \to C_{6k}(S^2)$ that takes a curve $A \in AlGen_k$ to the set of roots of its discriminant induces the homomorphism $br : \pi_1(AlGen_k) \to \pi_1(C_{6k}(S^2)) = \mathrm{H_{6k}}$.
- Choose the base points $C_0 \in AlGen_k$ and $D \in C_{6k}(S^2)$ for calculating $br(\pi_1(AlGen_k, C_0))$:
- Solid modification gives the transposition of discriminant roots, whereas dotted and waved ones leave the roots fixed.

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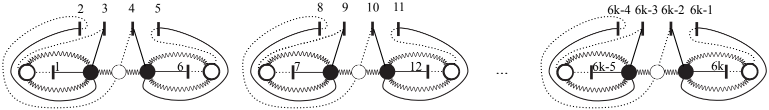

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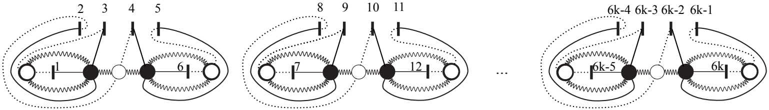
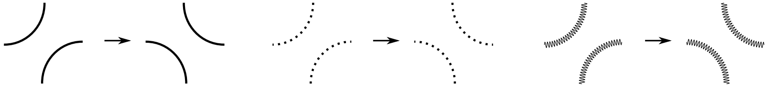


- Solid modification gives the transposition of discriminant roots, whereas dotted and waved ones leave the roots fixed.



Almost generic curves

Theorem

The image of $br : \pi_1(AlGen_k, C_0) \to \pi_1(C_{6k}(S^2), D)$ *is generated by the braids*
$\sigma_1^3, \sigma_2, \Delta_{1,3}^4, \sigma_3, \Delta_{4,6}^4, \sigma_4, \sigma_5^3, \Delta_{1,6}^{\sigma_1\sigma_2^2\sigma_1}, \sigma_5^{\sigma_6\sigma_7},$
$\sigma_7^3, \sigma_8, \Delta_{7,9}^4, \sigma_9, \Delta_{10,12}^4, \sigma_{10}, \sigma_{11}^3, \Delta_{7,12}^{\sigma_7\sigma_8^2\sigma_7}, \sigma_{11}^{\sigma_{12}\sigma_{13}}, \dots,$
$\sigma_{6k-5}^3, \sigma_{6k-4}, \Delta_{6k-5,6k-3}^4, \sigma_{6k-3}, \Delta_{6k-2,6k}^4, \sigma_{6k-2}, \sigma_{6k-1}^3, \Delta_{6k-5,6k}^{\sigma_{6k-5}\sigma_{6k-4}^2\sigma_{6k-5}},$
where $a^b = b^{-1}ab$ *and* $\Delta_{i,j} = (\sigma_i\sigma_{i+1}\dots\sigma_{j-1})(\sigma_i\sigma_{i+1}\dots\sigma_{j-2})\dots(\sigma_i\sigma_{i+1})\sigma_i$ *is the braid obtained by the rotation of the row of the lower ends of the strings by the angle* $-\pi$*.*

Nonsingular curves

Theorem

There is a homomorphism $\overline{br} : \pi_1(NSing_k) \to \mathrm{H_{6k}}/\{\sigma_1^3 = \sigma_{3\mathrm{i}\pm 1}^3 = 1\ \forall \mathrm{i} = 1, \dots, 2\mathrm{k}-1\}$*. Its image is generated by the cosets corresponding to the braids*

$\sigma_2, \sigma_3, \sigma_4, \Delta_{1,6}^{\sigma_1\sigma_2^2\sigma_1}, \sigma_5^{\sigma_6\sigma_7}$,

$\sigma_8, \sigma_9, \sigma_{10}, \Delta_{7,12}^{\sigma_7\sigma_8^2\sigma_7}, \sigma_{11}^{\sigma_{12}\sigma_{13}}, \dots$,

$\sigma_{6k-4}, \sigma_{6k-3}, \sigma_{6k-2}, \Delta_{6k-5,6k}^{\sigma_{6k-5}\sigma_{6k-4}^2\sigma_{6k-5}}$.

Thank you for your attention.